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FLOW PAST SYMMETRIC CONVEX PROFILES WITH OPEN WAKES

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SUMMARY

A fixed domain approach and a Baiocchi type transformation in conjunction with a modified Schwarz alternating iteration scheme are used to solve problems of flow past truncated convex shaped profiles between walls in a logarithmic hodograph plane. The flows are such that an open wake or cavity is formed behind the profile. The basic numerical scheme consists of the successive over-relaxation finite difference approach over the whole domain of the problem with the use of a projection operation over only part of the domain. The numerical results that are obtained using this approach for the cases of a truncated circular arc profile and a wedge profile are compared with published results and are found to be in good agreement.

KEY WORDS Cavities Fixed Domain Method Free Streamlines Schwarz Alternating Iteration Wakes

INTRODUCTION

Flow past a convex shaped profile and situated in a channel falls into the category of potential flow with a free streamline. Figure 1 shows such a case where the location of the free streamline is unknown. These two-dimensional, incompressible and inviscid flows are approximate models of the basic flows that occur in many practical engineering problems, for example the flow past bridge piers and channel constrictions, or a prototype under investigation in a wind tunnel. The objective is to provide basic potential-flow solutions to the problems and the determination of the location of the free streamline.

Analytical investigations of bodies moving (stationary) in stationary (moving) fluid with formation of wakes and regions of discontinuity behind them constitute one of the most difficult and complex branches of fluid mechanics. Failure of potential flow theory to predict such phenomenon in fluid flow, and even more its use in analysing such flow, had been accepted as a natural defect of the ideal fluid assumption for a long period of time. Introduction of an idealized inviscid flow model, with free streamlines as surfaces of discontinuity in 1869 by Kirchhoff, was the first major contribution to the subject. Kirchhoff used the conformal mapping technique that had been used by Helmholtz in 1868 for treating plane jets formed by free streamlines. Since the

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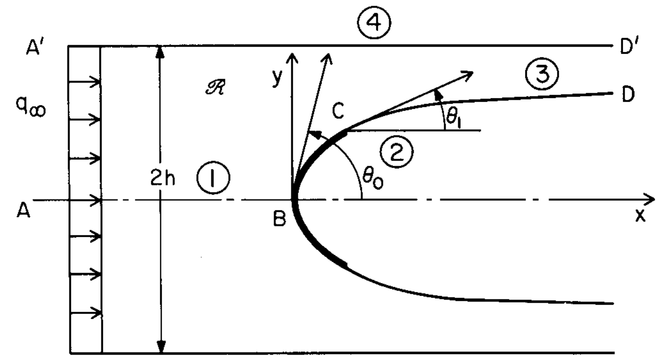

Figure 1. The profile in the physical plane

pioneering work of Kirchhoff, a number of potential flow models have been introduced to facilitate the mathematical analysis and to give the wake a definite configuration as an approximation to the inviscid outer flow. The purpose of these models is to bring the results of the potential theory of inviscid flows into better agreement with experimental measurements in fluids of small viscosity. The utility of the models is established by their capability of prediction of real fluid flow. Extensive reviews of the literature may be found in expositions by Birkhoff,[1] Birkhoff and Zarantonello,[2] Gilbarg,[3] Gurevich,[4] Robertson,[5] Sedov,[6] Robertson and Wislicenus,[7] Wehausen,[8] Woods[9] and Wu.[10,11]

The basic approach that will be used to solve the problem shown in Figure 1 is the fixed domain method in conjunction with the Baiocchi transformation.[12] This approach has had considerable success in solving various free and moving boundary problems.[13]

Several researchers have used the Baiocchi method to solve fluid flow problems. Their method requires a suitable Baiocchi transformation in the hodograph plane. The first researchers to approach fluid flow problems in this manner were Brezis and Stampacchia.[14] They reduced the study of a plane, steady, irrotational flow of a perfect and compressible fluid past a symmetric convex profile to one of a variational inequality in the hodograph plane. Brezis[15] goes into more detail concerning the theoretical aspects of the previous paper; see also Reference 16. Shimborsky[17] obtained an existence theorem for a subsonic symmetric potential flow of a compressible fluid in a symmetric plane channel with convex boundaries by solving a variational inequality in the hodograph plane. The method he used may be applied to obtain existence theorems for other plane flow problems. This work generalizes that of Brezis and Stampacchia.[14]

Brezis and Duvaut[18] extended the work of Brezis and Stampacchia[14] to flows with wakes. They, however, considered an incompressible fluid in their analysis. Working in the hodograph plane Brezis and Stampacchia[19] developed further properties of the solution using variational inequalities in the study of some two-dimensional subsonic flows past a given convex profile. For simplicity they confined themselves to the case of incompressible fluids. Roux[20] also studied the problem of subsonic compressible flow of a perfect fluid past a symmetric profile. He reduced the problem to solving a variational inequality with degenerate coefficients on the boundary of the domain. Special finite elements were used for the numerical computation. Ciavaldini *et al.*[21] were concerned with the determination of subcritical irrotational steady flows for a compressible inviscid fluid past a given profile in the physical plane. When the profile is convex and symmetric, their investigation of the stream functions in the hodograph plane leads to a linear variational inequality.[14] They give results using the finite element method as their numerical approximation; see also Reference 22.

Bourgat and Duvaut[23] presented numerical results for the elliptic variational inequality derived for the bi-dimensional steady flow of perfect fluid past a symmetric body formulated in the hodograph plane. With this approach they showed the existence of a one-parameter family of solutions. Furthermore, the study gave them a simple and effective method for calculating the solutions. Hummel[24,25] gives theoretical results for unsymmetric profiles in an incompressible infinite flow field using hodograph planes.

Tomarelli[26–28] investigated the case for a symmetric body located in a channel. He did not consider the situation with a wake. Dormiani and Bruch[29] and Bruch and Dormiani[30] extended the work of Tomarelli by investigating cases of steady flow past a symmetric body located between walls and with a wake. Ciavaldini *et al.*[31] studied stationary irrotational plane flows of an incompressible inviscid fluid past a profile between porous walls. They looked for the stream function in a functional form, especially convenient for a numerical approach. Using a finite element method of order one they solved this new problem in the physical plane and gave some numerical results. In all the previously referenced papers when the term hodograph plane is used it refers to the plane obtained by taking the natural log of the conjugate of the complex velocity plane and then multiplying by $(-i)$.

The problem presented herein has features which are different from the previously mentioned problems. Although the fixed domain approach and a Baiocchi type transformation are applicable, they will be used in conjunction with a modified Schwarz alternating iteration scheme. This latter scheme is convenient since the Baiocchi type transformation is not applicable over the entire solution domain as it was in the previous problems discussed. The numerical results that are obtained using this technique for flow past a profile between walls will be compared with those of Davis in Reference 32 for a truncated circular arc profile and those of Street[33] for a wedge profile.

FORMULATION OF THE PROBLEM

Consider a profile with the shape of a convex curve contained in a channel which can be visualized as an infinitely wide rectangular conduit of height $2h$ (see Figure 1). The area behind the wedge and enclosed by the free streamlines can be considered either as a region filled with stagnant fluid (for example water) or as a cavity filled with air and fluid vapour (e.g. water vapour). The flow field has a pair of free streamlines on which the pressure and velocity are constants. Take these to be equal to p_c and q_c, respectively. The channel height, $2h$, the velocity on the boundary of the cavity, q_c, the profile shape and separation angle are assumed to be known, whereas the upstream velocity in the channel, q_∞, and the free streamline location are to be found.

The basic relations are established from the continuity equation and irrotationality condition. These equations in differential form for the flow field are

$$\operatorname{div} \mathbf{q} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0, \tag{1}$$

$$\operatorname{curl} \mathbf{q} = \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0, \tag{2}$$

where q_1 and q_2 are the fluid velocities in the x and y co-ordinate directions, respectively. The problem is formulated and solved in terms of the stream function ψ, which is defined so that

$$q_1 = \frac{\partial \psi}{\partial y} \quad \text{and} \quad q_2 = -\frac{\partial \psi}{\partial x}. \tag{3}$$

Because of symmetry the region under consideration, $\mathscr{R}$, is bounded between the axis of symmetry, AB, half of the profile, BC, the free streamline, CD, and the wall of the channel, D'A'. In this region the stream function satisfies the continuity equation identically and the irrotationality condition gives Laplace's equation

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0, \quad \text{in } \mathscr{R}. \tag{4}$$

The boundary ABCD forms the $\psi = 0$ streamline and $\psi = q_\infty h$ on the wall D'A'. The downstream jet half-width d_c is found from the conservation of mass relation

$$q_c d_c = q_\infty h \tag{5}$$

and q_c may be set to unity without any loss of generality. Therefore, the mathematical formulation of the problem in the physical plane becomes: find $\psi(x, y)$ such that

$$\nabla^2\psi = 0, \quad \text{in } \mathscr{R} \tag{6a}$$

and

$$\psi(x, y) = 0, \quad \text{on} \quad \text{ABCD}, \tag{6b}$$

$$\psi = q_\infty h, \quad \text{on} \quad \text{D'A'}, \tag{6c}$$

$$\lim_{x\to-\infty} \psi(x, y) = yq_\infty, \quad \text{on} \quad \text{AA'}, \tag{6d}$$

$$\lim_{x\to+\infty} \psi(x, y) = [y - (h - d_c)]q_c, \quad \text{on} \quad \text{DD'}, \tag{6e}$$

$$|\text{grad}\,\psi| = q_c \quad \text{on} \quad \text{CD}. \tag{6f}$$

Note that the location of the free streamline is unknown in advance.

TRANSFORMATION TO A LOGARITHMIC HODOGRAPH PLANE

The flow region shown in Figure 1 may be mapped conformally onto a region of the logarithmic hodograph plane. From the flow pattern it may be seen that the velocity, q, at any point, is always less than or equal to q_c. At the stagnation point B, $q = 0$; far upstream, $q = q_\infty$; and on the free streamline, $q = q_c$.

The transformation $(x, y) \to (\theta, \sigma)$, where θ is the polar angle of velocity and $\sigma = -\ln(q/q_c)$ maps the problem in the physical plane onto the logarithmic hodograph plane. Under the conformal mapping, the values of the harmonic function ψ are unchanged on the boundaries of the region.

The form of the governing equation, as is shown below, also remains the same. It is necessary to find the differential expression for the equation $\nabla^2\psi(x, y) = 0$, which is expressed in terms of the co-ordinates, x and y, in terms of the new variables, θ and σ. In the transformation $(x, y) \to (\theta, \sigma)$ use is made of

$$\mathrm{d}x + \mathrm{i}\,\mathrm{d}y = \frac{\mathrm{e}^{\sigma+\mathrm{i}\theta}}{q_c}(\mathrm{d}\phi + \mathrm{i}\,\mathrm{d}\psi). \tag{7}$$

Since this equation is in terms of total differentials, it follows that

$$\frac{\partial}{\partial\sigma}\left[\frac{\mathrm{e}^{\sigma+\mathrm{i}\theta}}{q_c}\left(\frac{\partial\phi}{\partial\theta} + \mathrm{i}\frac{\partial\psi}{\partial\theta}\right)\right] = \frac{\partial}{\partial\theta}\left[\frac{\mathrm{e}^{\sigma+\mathrm{i}\theta}}{q_c}\left(\frac{\partial\phi}{\partial\sigma} + \mathrm{i}\frac{\partial\psi}{\partial\sigma}\right)\right]. \tag{8}$$

Differentiating and simplifying gives

$$\frac{\partial \phi}{\partial \sigma}=\frac{\partial \psi}{\partial \theta}, \quad -\frac{\partial \psi}{\partial \sigma}=\frac{\partial \phi}{\partial \theta}. \tag{9}$$

Eliminating ϕ from equation (9) yields

$$\frac{\partial^2 \psi}{\partial \theta^2}+\frac{\partial^2 \psi}{\partial \sigma^2}=\nabla^2\psi(\theta,\sigma)=0. \tag{10}$$

The governing equation and boundary conditions are, therefore

$$\nabla^2\psi(\theta,\sigma)=0, \quad \text{in} \quad R, \tag{11a}$$

$$\psi(\theta,0)=0, \quad 0\leqslant\theta\leqslant\theta_1, \tag{11b}$$

$$\psi(\theta,\sigma)=0, \quad \text{on} \quad \Gamma, \tag{11c}$$

$$\frac{\partial \psi}{\partial \sigma}=-\frac{q_c e^{-\sigma}\tilde{R}(\theta)}{1+l'^2}, \quad \text{on} \quad \Gamma, \tag{11d}$$

$$\frac{\partial \psi}{\partial \theta}=-l'\frac{\partial \psi}{\partial \sigma}=\frac{q_c e^{-\sigma}\tilde{R}(\theta)l'}{1+l'^2}, \quad \text{on} \quad \Gamma, \tag{11e}$$

$$\psi(0,\sigma)=\begin{cases} hq_\infty, & 0\leqslant\sigma\leqslant\sigma_\infty=-\ln\dfrac{q_\infty}{q_c}, \\ 0, & \sigma>\sigma_\infty, \end{cases} \tag{11f}$$

where R is the image of region $\mathscr{R}$ under the transformation, and Γ is the representation of the profile in the logarithmic hodograph plane (see Figure 2). On Γ, $\sigma=l(\theta)$. Note that the location of Γ and the point $(0,\sigma_\infty)$ are unknown *a priori.*

The region R of the problem in the logarithmic hodograph plane is next divided into two overlapping regions, R_ψ and R_p (see Figure 3) such that $R=R_\psi\cup R_p$ and

$$R_\psi=\{(\theta,\sigma)|0<\theta\leqslant\theta_1,\sigma>0\}\cup\{(\theta,\sigma)|\theta_1<\theta<\bar{\alpha},\sigma>l(\theta)\},$$

$$R_p=\{(\theta,\sigma)|\theta_1<\theta<\theta_0,\sigma>l(\theta)\},$$

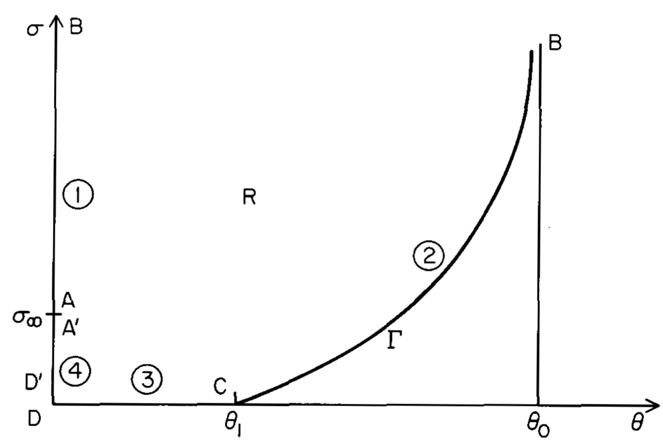

Figure 2. The problem in the logarithmic hodograph plane ((θ,σ) plane)

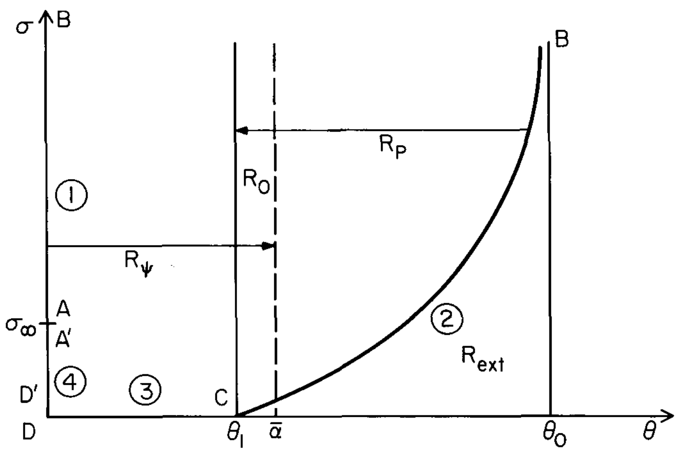

Figure 3. The problem in the logarithmic hodograph plane ((θ, σ) plane) with the overlapping regions

where θ_0 is the value of θ at the stagnation point, θ_1 is the value at the detachment point of the cavity boundary from the profile and $\bar{\alpha}$ is an angle satisfying $\theta_1 < \bar{\alpha} < \theta_0$. Note that the overlapped region is

$$R_0 = R_\psi \cap R_p.$$

Define an integrated stream function by using the Baiocchi type transformation

$$u(\theta, \sigma) = \frac{e^{-\sigma}}{q_c} \int_{l(\theta)}^{\sigma} e^{\tau} \psi(\theta, \tau) \, d\tau, \tag{12}$$

on the region R_p. Note that $u > 0$ in R_p. Since $\psi > 0$ thereafter differentiating,

$$\nabla^2 u(\theta, \sigma) = \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \sigma^2} = -\tilde{R}(\theta) e^{-\sigma} \quad \text{in} \quad R_p \tag{13}$$

where

$$\tilde{R}(\theta) = -\left[\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2\right]^{1/2} = -[X'(\theta)^2 + Y'(\theta)^2]^{1/2},$$

the algebraic radius of curvature, in which the co-ordinates $X(\theta)$ and $Y(\theta)$ represent the surface of the profile in terms of the parameter θ, the angle between the tangent to the curve and the x-axis. Next, the dependent variable u is continuously extended across the free surface Γ, on which $u = 0$, into

$$R_{ext} = \{(\theta, \sigma) | \theta_1 < \theta < \theta_0, 0 < \sigma < l(\theta)\},$$

such that u is zero in R_{ext}. Let $R_u = R_p + R_{ext} + \Gamma$. Then the following complementarity system is defined in R_u: since

$$u \geqslant 0, [-\nabla^2 u(\theta. \sigma) - \tilde{R}(\theta) e^{-\sigma}] \geqslant 0,$$

then

$$u[\nabla^2 u(\theta, \sigma) + \tilde{R}(\theta) e^{-\sigma}] = 0. \tag{14}$$

Boundary conditions for the overlapped region, R_0, are needed. On the line

$$\Gamma_1 = \{(\theta, \sigma) | \theta = \theta_1, \quad \sigma > 0\} \tag{15}$$

using the definition of u (equation (12)) define

$$u(\theta_1, \sigma) = \frac{e^{-\sigma}}{q_c} \int_0^\sigma e^\tau \psi(\theta_1, \tau)\, d\tau, \quad \text{on} \quad \Gamma_1. \tag{16}$$

This boundary condition is used on Γ_1 for the region R_u. From the definition of u, u_σ can be found, and thus

$$u + u_\sigma = \frac{\psi(\theta, \sigma)}{q_c}. \tag{17}$$

Therefore, the boundary condition on

$$\Gamma_2^1 = \{(\theta, \sigma) | \theta = \bar{\alpha}, \quad \sigma \geqslant l(\theta)\}, \tag{18}$$

for the region R_ψ is equation (17). Furthermore, $\psi(\theta, \sigma) = 0$ on Γ_2^2, where $\Gamma_2^2 = \{(\theta, \sigma) | \theta_1 \leqslant \theta \leqslant \bar{\alpha}, \sigma = l(\theta)\}$. Set $\Gamma_2 = \Gamma_2^1 + \Gamma_2^2$. Hence, the problem can be stated in the two overlapping regions as follows.

On region R_ψ:

$$\nabla^2 \psi(\theta, \sigma) = 0, \quad \text{in} \quad R_\psi, \tag{19a}$$

$$\psi(\theta, 0) = 0, \quad 0 \leqslant \theta \leqslant \theta_1, \tag{19b}$$

$$\psi(0, \sigma) = \begin{cases} hq_\infty, & 0 \leqslant \sigma \leqslant \sigma_\infty, \quad (19c) \\ 0, & \sigma > \sigma_\infty, \quad (19d) \end{cases}$$

$$\psi(\theta, \sigma) = q_c(u + u_\sigma), \quad \text{on} \quad \Gamma_2^1, \tag{19e}$$

$$\psi(\theta, \sigma) = 0, \quad \text{on} \quad \Gamma_2^2; \tag{19f}$$

and on region R_u:

$$\nabla^2 u(\theta, \sigma) = -\tilde{R}(\theta) e^{-\sigma} \chi_{R_p}, \quad \text{in} \quad R_u, \tag{20a}$$

$$u(\theta, 0) = 0, \quad \theta_1 \leqslant \theta \leqslant \theta_0, \tag{20b}$$

$$u(\theta_0, \sigma) = 0, \quad 0 \leqslant \sigma, \tag{20c}$$

$$u(\theta_1, \sigma) = \frac{e^{-\sigma}}{q_c} \int_0^\sigma e^\tau \psi(\theta_1, \tau)\, d\tau, \quad \text{on} \quad \Gamma_1, \tag{20d}$$

where χ_{R_p} is the characteristic function defined by $\chi_{R_p} = 1$ in R_p and $\chi_{R_p} = 0$ otherwise.

ALTERNATING ITERATION SOLUTION

In this section, an iteration solution scheme is described upon which the numerical method given later is based. The method is called the Schwarz alternating procedure. A proof and justification that the method is valid in the case considered is not given. However, the method will be used none the less, as if it were applicable for computational purposes; see Reference 34 for details.

The scheme permits one to solve certain boundary value problems whenever the region of the problem consists of two overlapping regions with smooth boundaries. This is the nature of the problem formulated in the previous section where Γ_1 and Γ_2 are part of the boundary of the overlapped region. The procedure can be described as follows.

By defining an arbitrary continuous boundary condition on Γ_2 the problem for ψ in the region R_ψ is solved. The solution provides values on Γ_1 which together with the other prescribed boundary conditions furnish continuous boundary conditions for u for the region R_u. This solution determines the values of u on Γ_2 and thus provides boundary conditions for ψ on Γ_2, and with prescribed boundary conditions on the other boundaries of R_ψ, the problem can be solved for ψ, and after that u and the region R_u can again be considered. Continuing alternately in this way and switching between regions R_ψ and R_u generates a sequence which it is hoped converges to functions which are identical in the overlapped region, R_0.

CO-ORDINATE TRANSFORMATIONS

The expressions describing the co-ordinates of the physical plane in terms of the co-ordinates of the logarithmic hodograph plane, i.e. $x = x(\theta, \sigma)$ and $y = y(\theta, \sigma)$ are needed for calculating the co-ordinates of the wake boundary. These expressions are stated in this section.

The desired equations are

$$\mathrm{d}x = \frac{\mathrm{e}^\sigma}{q_c}\left(\frac{\partial\psi}{\partial\sigma}\cos\theta + \frac{\partial\psi}{\partial\theta}\sin\theta\right)\mathrm{d}\theta + \frac{\mathrm{e}^\sigma}{q_c}\left(\frac{\partial\psi}{\partial\sigma}\sin\theta - \frac{\partial\psi}{\partial\theta}\cos\theta\right)\mathrm{d}\sigma \tag{21a}$$

and

$$\mathrm{d}y = \frac{\mathrm{e}^\sigma}{q_c}\left(\frac{\partial\psi}{\partial\sigma}\sin\theta - \frac{\partial\psi}{\partial\theta}\cos\theta\right)\mathrm{d}\theta - \frac{\mathrm{e}^\sigma}{q_c}\left(\frac{\partial\psi}{\partial\sigma}\cos\theta + \frac{\partial\psi}{\partial\theta}\sin\theta\right)\mathrm{d}\sigma. \tag{21b}$$

On the boundary of the wake the velocity is constant and is equal to q_c; therefore $\sigma = 0$ and $\mathrm{d}\sigma = 0$. Furthermore $\psi = \text{const.}$; hence $\partial\psi/\partial\theta = 0$, and equations (21) reduce to

$$\mathrm{d}x = \frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\cos\theta\,\mathrm{d}\theta, 0 \leqslant \theta \leqslant \theta_1, \quad \sigma = 0 \tag{22a}$$

and

$$\mathrm{d}y = \frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\sin\theta\,\mathrm{d}\theta, 0 \leqslant \theta \leqslant \theta_1, \quad \sigma = 0 \tag{22b}$$

NUMERICAL PROCEDURE

The problem posed in the logarithmic hodograph plane by equations (19) and (20) is formulated in a region which is unbounded in the positive σ-direction. For numerical computations the region will be truncated. Toward this end choose a σ_u which is sufficiently large so that for all practical purposes the value of ψ and u for $\sigma > \sigma_u$ is approximately zero. Note that since the function defining σ is logarithmic, $\sigma = -\ln(q/q_c)$, and u is weighted by an exponential function, the truncation has little or no effect on the numerical results. Hence, σ_u provides an upper bound for $R_u \cup R_\psi$. The solution algorithm is a finite difference successive over-relaxation scheme for both u and ψ with projection for the u-problem only. A grid of mesh points is superimposed on the bounded region, where each node is specified by row i and column j. Therefore, the field equation for ψ, equation (19a), can be written as the following difference equation:

$$\psi_{i,j}^{(n+1/2)} = \frac{\alpha\beta(\Delta\theta)^2(\Delta\sigma)^2}{2[\alpha(\Delta\theta)^2+\beta(\Delta\sigma)^2]}\left[\frac{2}{(1+\alpha)(\Delta\theta)^2}\psi_{i,j-1}^{(n+1)} + \frac{2}{\alpha(1+\alpha)(\Delta\theta)^2}\psi_{i,j+1}^{(n)} + \frac{2}{(1+\beta)(\Delta\sigma)^2}\psi_{i-1,j}^{(n+1)} + \frac{2}{\beta(1+\beta)(\Delta\sigma)^2}\psi_{i+1,j}^{(n)}\right], \tag{23a}$$

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)} + \omega(\psi_{i,j}^{(n+1/2)} - \psi_{i,j}^{(n)}), \tag{23b}$$

where $\Delta\theta$ and $\Delta\sigma$ are the spacings in θ and σ directions, respectively, α and β provide for unequal divisions for mesh points, ω is the over-relaxation parameter and $\psi_{i,j}^{(n)}$ is the value of ψ at node i, j for the nth iteration. Similarly, for u in the region R_u (see equation (20a)):

$$u_{i,j}^{(n+1/2)} = \frac{\alpha\beta(\Delta\theta)^2(\Delta\sigma)^2}{2[\alpha(\Delta\theta)^2+\beta(\Delta\sigma)^2]}\left[\frac{2}{(1+\alpha)(\Delta\theta)^2}u_{i,j-1}^{(n+1)} + \frac{2}{\alpha(1+\alpha)(\Delta\theta)^2}u_{i,j+1}^{(n)} + \frac{2}{(1+\beta)(\Delta\sigma)^2}u_{i-1,j}^{(n+1)} + \frac{2}{\beta(1+\beta)(\Delta\sigma)^2}u_{i+1,j}^{(n)} + \tilde{R}(\theta_j)e^{-\sigma_i}\right], \tag{24a}$$

$$u_{i,j}^{(n+1)} = \max\{0, u_{i,j}^{(n)} + \omega(u_{i,j}^{(n+1/2)} - u_{i,j}^{(n)})\}, \tag{24b}$$

where $u_{i,j}^{(n)}$ is the value of u at node i, j for the nth iteration. These iterations are stopped when

$$\max_{i,j}|\psi_{i,j}^{(n+1)} - \psi_{i,j}^{(n)}| < \varepsilon \tag{25a}$$

and

$$\max_{i,j}|u_{i,j}^{(n+1)} - u_{i,j}^{(n)}| < \varepsilon, \tag{25b}$$

where ε is some fixed positive constant.

The values of the boundary conditions that are given, such as equations (19b)–(19f) and equations (20b) and (20d), are directly substituted into the difference equations. Note that on the boundary

$$\Gamma_3 = \{(\theta,\sigma) | 0 \leqslant \theta \leqslant \theta_1, \quad \sigma = \sigma_u\},$$

which corresponds to the stagnation point in the physical plane, $\psi \approx 0$, for the region R_ψ. On the boundary

$$\Gamma_4 = \{(\theta,\sigma) | \theta_1 \leqslant \theta \leqslant \theta_0, \quad \sigma = \sigma_u\},$$

$\psi \approx 0$ also, and equation (17) yields

$$u + u_\sigma = 0$$

which provides the appropriate boundary condition.

The boundary conditions for the overlapped region are given as follows. From the definition of region R_ψ, for numerical analysis the value of $\bar{\alpha}$ is chosen so that the first column on interior mesh points of region R_u coincides with Γ_2^1 and forms the boundary for the R_ψ region. The values of ψ at the mesh points on Γ_2^1 are calculated by using equation (19e), in which u_σ is approximated by a central difference expression; therefore

$$\psi(\bar{\alpha}, \sigma_i) = q_c\left[u(\bar{\alpha}, \sigma_i) + \frac{u(\bar{\alpha}, \sigma_{i+1}) - u(\bar{\alpha}, \sigma_{i-1})}{2\Delta\sigma}\right]. \tag{26}$$

It should be noted that for the numerical computations the first mesh point on Γ_2^1 at $(\bar{\alpha}, \sigma_i)$, $i = 2$ must be such that $\sigma_2 \geqslant l(\bar{\alpha})$.

On the other hand the last column of interior mesh points in region R_ψ, which are on the line Γ_1, forms the boundary of the region R_u, and equation (20d) is used to calculate the boundary condition. The integral in this equation is approximated by using a mid-point formula; hence

$$u(\theta_1,\sigma_i)=\frac{e^{-\sigma_i}}{q_c}\frac{\Delta\sigma}{2}[e^{\sigma_{i-1}}\psi(\theta_1,\sigma_{i-1})+e^{\sigma_i}\psi(\theta_1,\sigma_i)]+u(\theta_1,\sigma_{i-1}). \tag{27}$$

The iteration sequence is started by setting the boundary conditions for ψ in the region R_ψ, and a zero initial guess for the interior $\psi_{i,j}^{(0)}$. Then using equations (23), starting from the lower left interior point and moving to the right along mesh points and upwards until all the interior points in R_ψ are covered, the $\psi_{i,j}^{(1)}$ are obtained. The next step is to set the boundary conditions for R_u making use of the newly calculated values for the overlapped region, a zero initial guess for the interior $u_{i,j}^{(0)}$, using equations (24), and again starting from the lower left point in the region R_u and moving to the right and upwards in this region until the entire mesh is covered; this provides new values for points on the overlapped region and hence new boundary conditions for ψ in region R_ψ.

This alternate sweeping of the two regions continues until the conditions (25) are satisfied. Since numerical values for u inside the free boundary, Γ, are non-zero and those on the boundary and outside of it are zero, the zero points bordering non-zero points in the R_u region determine this free boundary.

The velocity on the boundary of the cavity, q_c, is assumed to be known but, as stated before, the upstream velocity q_∞ is, like the free boundary, unknown *a priori* and is to be found as part of the solution. Therefore, different values for σ_∞, where $\sigma_\infty=-\ln(q_\infty/q_c)$, are used until the best one is found. The calculation sequence is that of assuming the mesh points on the boundary

$$\Gamma_5=\{(\theta,\sigma)|\theta=0,\quad 0<\sigma<\sigma_u\},$$

starting from the point with minimum σ and going upwards. For each assumed σ_∞ the alternating iteration sequence described above is performed and co-ordinates of the wake, using equations (22), are calculated.

For calculating co-ordinates of the boundary of the wake equations (22) are integrated between two adjacent mesh points using the trapezoidal rule, which yields

$$x_{j-1}=x_j+\frac{\Delta\theta}{2q_c}\left(\left.\frac{\partial\psi}{\partial\sigma}\right|_j\cos\theta_j+\left.\frac{\partial\psi}{\partial\sigma}\right|_{j-1}\cos\theta_{j-1}\right), \tag{28a}$$

$$y_{j-1}=y_j+\frac{\Delta\theta}{2q_c}\left(\left.\frac{\partial\psi}{\partial\sigma}\right|_j\sin\theta_j+\left.\frac{\partial\psi}{\partial\sigma}\right|_{j-1}\sin\theta_{j-1}\right), \tag{28b}$$

where $(\partial\psi/\partial\sigma)|_j$ is approximated by its forward difference expression

$$\left.\frac{\partial\psi}{\partial\sigma}\right|_j=\frac{1}{2\,\Delta\sigma}[-3\psi(\theta_j,\sigma_i)+4\psi(\theta_j,\sigma_{i+1})-\psi(\theta_j,\sigma_{i+2})].$$

Once the co-ordinates of the wake are determined, the cavity distance, d_c, which is the distance between the boundary of the cavity and the wall at infinity (see Figure 1) is calculated. Then from equation (5) the upstream velocity q_∞, or consequently σ_∞, is calculated and is compared to the assumed value of σ_∞. The mesh point corresponding to the minimum difference between the calculated upstream velocity and assumed upstream velocity is chosen for the desired value for σ_∞. It is evident that the finer the mesh points are on boundary Γ_5 the better is the accuracy in the determination of σ_∞.

COMPUTATIONAL RESULTS

Although the numerical method discussed in the previous section is general and can be carried out for any convex symmetric profile, results are given for free streamline flows past a truncated circular profile since there are previous calculations close to this case in the literature for comparison purposes.

Figure 4 shows the results for an open profile which has the shape of an arc of a circle and for which the free streamline leaves the profile at 60° from the stagnation point ($\theta_1 = 30°$). The profile is located between walls each having a distance $h = 4{\cdot}0$ from the axis of symmetry. To obtain the results for the case where there are no walls the wall distance is increased sufficiently so that the walls would have no effect on the solution. For this case h was taken to be 25. This case is also shown in Figure 4 along with the results given by Davis in Reference 32. As can be seen from the Figure, the shapes and locations of the free streamlines are in appropriate agreement. The velocity on the boundary of the cavity was assumed to be $q_c = 1$, the over-relaxation parameter was taken to be $\omega = 1{\cdot}6$, the stopping criterion for equations (25) was $\varepsilon = 10^{-4}, h = 4{\cdot}0, \sigma_u = 4{\cdot}621$, the number of divisions in the θ and σ directions was 20 with variable $\Delta\theta$ and $\Delta\sigma$. The upstream uniform fluid velocity was computed to be $q_\infty = 0{\cdot}745$. For the case when $h = 25{\cdot}0$, the previous parameters were also used except that the numbers of divisions in the θ and σ directions were 20 and 40, respectively, with variable $\Delta\theta$ and $\Delta\sigma$. The upstream uniform velocity was computed to be $q_\infty = 0{\cdot}918$.

Another limiting case of the theory is the situation where the profile is a wedge, as shown in Figure 5. This is the case solved by Street.[33] The logarithmic hodograph plane for this case is shown in Figure 6.

As an example, numerical computations were carried out for a wedge with a half angle of $\bar{\beta} = 30°$, side length of $c = 3{\cdot}15$, and wall distance of $2h = 8{\cdot}0$; the velocity on the boundary of the cavity was assumed to be $q_c = 1$. The logarithmic hodograph was truncated in the σ direction at $\sigma_u = 4{\cdot}621$. The over-relaxation parameter was chosen to be $\omega = 1{\cdot}6$; the criterion for stopping the iteration was $\varepsilon = 10^{-5}$; and the number of divisions in the θ and σ directions was 40, i.e. $\Delta\theta = (\pi/6)/40$ and $\Delta\sigma = (4{\cdot}621)/40$. The upstream velocity was found to be $q_\infty = 0{\cdot}501$ and the co-ordinates of the boundary of the wake were calculated and are plotted in Figure 7. Also

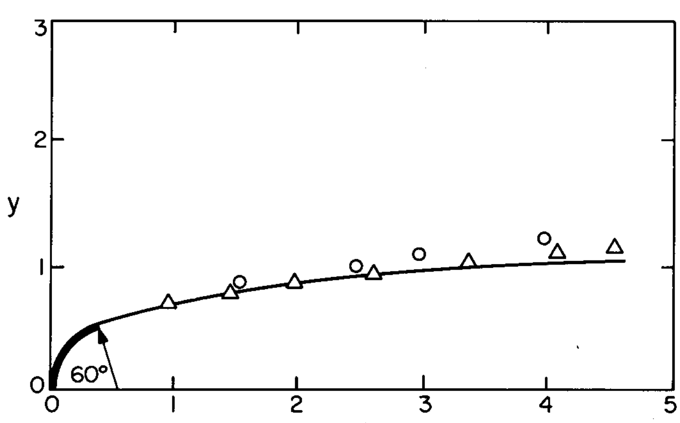

Figure 4. Results for a truncated circular arc profile: ——— present solution with walls ($h = 4{\cdot}0$); △ present solution with walls ($h = 25{\cdot}0$); ○ Reference 32

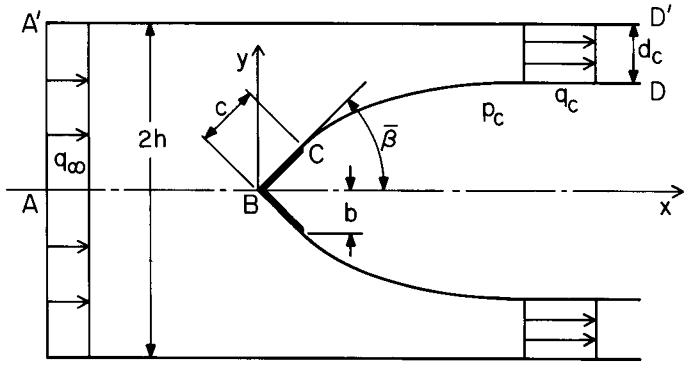

Figure 5. Flow past a wedge in a channel

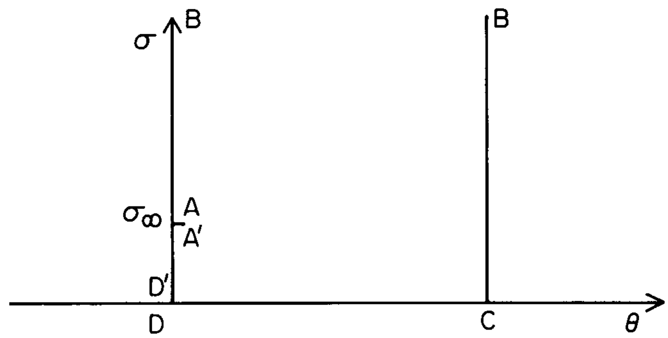

Figure 6. Logarithmic hodograph plane ((θ, σ) plane) for a wedge in a channel

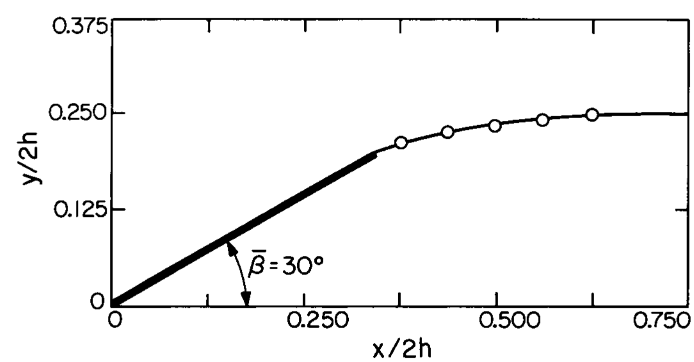

Figure 7. Location of the free streamline:——— present solution; ○ Street

shown on the Figure are results given by Street.[33] Street's solution for the wedge profile is given in terms of infinite integrals of elementary functions, from which numerical results can easily be obtained. He also gives some of the results in graphical form. As can be seen the two sets of results are in agreement.

CONCLUSIONS

Problems of flow past truncated convex shaped profiles between walls have been solved in the logarithmic hodograph plane, the (θ, σ) plane, using a fixed domain approach and a Baiocchi type transformation in conjunction with a modified Schwarz alternating iteration scheme. The numerical scheme used in the iteration scheme was the successive over-relaxation finite difference approach for both the ψ-problem and the u-problem, whereas the projection operation was used only on the u-problem. The fundamental difference between this problem and the problems mentioned in the Introduction which were solved using the integral transform approach is that there are two regions, instead of one, to contend with in the logarithmic hodograph plane.

The numerical algorithm that has been derived is simple and efficient and, as seen from the comparisons of results, gives accurate solutions. Thus, the solution approach that has been derived can be applied to general truncated convex shaped profiles between walls whose profile shape is *a priori* known, as opposed to other schemes where the profiles are obtained as part of the solution. Furthermore, the numerical scheme gives the velocity along the profile which is the curve Γ in the (θ, σ)-plane as part of the solution. This is the line that separates the region where $u > 0$ from that where $u = 0$. This free boundary problem is different from other free boundary type problems in that the free streamline CD is a horizontal line in the (θ, σ)-plane, whereas the velocity distribution on BC becomes that on Γ, the boundary sought in the (θ, σ)-plane.

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